Fermat-type equations of signature $(13, 13, p)$ via Hilbert cuspforms.

Luis Dieulefait and Nuno Freitas

January 31, 2012

Abstract

In this paper we prove that equations of the form $x^{13}+y^{13} = Cz^p$ have no non-trivial primitive solutions (a, b, c) such that $13 \nmid c$ if $p > 4992539$ for an infinite family of values for C. Our method consists in relating a solution (a, b, c) to the previous equation to a solution (a, b, c_1) of another Diophantine equation with coefficients in $\mathbb{Q}(\sqrt{13})$. We then construct Frey-curves associated with (a, b, c_1) and we prove modularity of them in order to apply the modular approach via Hilbert cusp forms over $\mathbb{Q}(\sqrt{13})$. We also prove a modularity result for elliptic curves over totally real cyclic number fields of interest by itself.

1 Introduction

Since the proof of Fermat's Last Theorem by Wiles [27] the modular approach to Diophantine equations has been popularized and achieved great success in solving equations that previously seemed intractable. In order to attack the generalized Fermat equation $Ax^p + By^q = Cz^r$, where $1/p+1/q+1/r < 1$ the initial strategy of Frey, Hellegouarch, Serre, Ribet and Wiles was strengthened and many particular cases, including infinite families, were solved. An important progress is the fact (as a consequence of the work of Darmon-Granville [9]) that for a fixed triple (p, q, r) there exists only a finite number of solutions such that $(x, y, z) = 1$. Another important progress was the work of Ellenberg on the representations attached to $\mathbb{Q}$-curves which allowed to use a special type of elliptic curves over number fields ($\mathbb{Q}$-curves) to attack Diophantine equations over the rationals. In particular, Ellenberg solved the equation $A^4 + B^2 = C^p$ (see [13]). For a summary of known results on the equation $x^p + y^q = z^r$ see the introduction in [6].

A particularly important subfamily of the generalized Fermat equation are the equations of signature (r, r, p), that is, $Ax^r + By^r = Cz^p$ with r a fixed prime. In this direction there is work for $(3, 3, p)$ by Kraus [20], Bruin [5], Chen-Siksek [7] and Dahmen [8]; for $(5, 5, p)$ by Billerey [1], Dieulefait-Billerey [2] and from the authors [12]; for $(7, 7, p)$ from the second author (currently in preliminary version). In this paper we will go further into this family of equations and we will use a generalized version of the classical modular approach to study equations of the form

$$x^{13} + y^{13} = Cz^p. \tag{1}$$

Let (a,b,c) be a triple of integers such that $a^{13}+b^{13}=Cc^p$. We say that it is a *primitive* solution if $(a,b)=1$ and we will say that it is a *trivial* solution if $abc=0$. Following the terminology introduced by Sophie Germain in her work on the FLT we will divide solutions to (1) into two cases

Definition 1.1 *A primitive solution (a,b,c) of $x^r+y^r=Cz^p$ is called a first case solution if r do not divide c, and a second case solution otherwise.*

The strategy we will use goes as follows: we first relate a possible non-trivial primitive solution (a,b,c) of (1) to a non-trivial primitive solution (a,b,c_1) of another Diophantine equation with coefficients in $\mathbb{Q}(\sqrt{13})$. Secondly we attach to the latter solution a Frey-Hellegouarch-curve over $\mathbb{Q}(\sqrt{13})$ that is not a $\mathbb{Q}$-curve. Then using modularity results from Skinner-Wiles and Kisin we prove modularity of our F-H-curves. With modularity established, from the level lowering results for Hilbert modular forms it will follow that the existence of the solution (a,b,c_1) implies a congruence between two Galois representations. One of the representations is attached to our Frey-curves and the other to a Hilbert newform of a certain level (not depending on c_1) and parallel weight $(2,2)$. Finally we will show that, in some cases, this congruence can not hold and so the equation over $\mathbb{Q}(\sqrt{13})$ can not have non-trivial primitive solutions and consequently neither (1). The main result in this paper is the following theorem:

Theorem 1.2 *Let $d=3,5,7$ or 11 and γ be an integer divisible only by primes $l\not\equiv 1$ (mod 13). If $p>4992539$ is a prime, then:*

(I) The equation $x^{13}+y^{13}=d\gamma z^p$ has no non-trivial primitive first case solutions.

(II) The equation $x^{26}+y^{26}=10\gamma z^p$ has no primitive non-trivial solutions.

In what follows we will first prove part (I) of Theorem 1.2 and in the end we will explain the small tweak needed to conclude part (II). Observe that in part (II) replacing 10 by twice d for $d=3,7,11$ the statement is also true but trivial, because the left-hand side is a sum of two relatively prime squares.

We thank John Cremona for providing us a list of elliptic curves that was useful to test our strategy. We also want to thank John Voight for computing a list of Hilbert modular forms that was fundamental to finish this work.

2 Relating two Diophantine equations

In this section we will relate a solution of (1) to a solution of a Diophantine equation with coefficients in $\mathbb{Q}(\sqrt{13})$. In order to do that we will need a few properties on the factors of $x^{13}+y^{13}$ over the cyclotomic field $\mathbb{Q}(\zeta_{13})$. Since these properties are not exclusive of degree 13 we will prove them in general. Observe that if r is a prime then

$$x^r+y^r=(x+y)\phi_r(x,y)$$

where

$$\phi_r(x,y) = \prod_{i=0}^{r-1} (-1)^i x^{r-1-i} y^i.$$

Let $\zeta = \zeta_r$ be an r-root of unity and consider the decomposition over the cyclotomic field $\mathbb{Q}(\zeta)$

$$\phi_r(x,y) = \prod_{i=1}^{r-1} (x + \zeta^i y). \tag{2}$$

Proposition 2.1 *Let $\mathfrak{P}_r$ be the prime in $\mathbb{Q}(\zeta)$ above the rational prime r and suppose that $(a,b)=1$. Then, any two distinct factors $a+\zeta^i b$ and $a+\zeta^j b$ in the factorization of $\phi_r(a,b)$ are coprime outside $\mathfrak{P}_r$. Furthermore, if $r \mid a+b$ then $\nu_{\mathfrak{P}_r}(a+\zeta^i b)=1$ for all i.*

Proof: Suppose that $(a,b)=1$. Let $\mathfrak{P}$ be a prime in $\mathbb{Q}(\zeta)$ above $p \in \mathbb{Q}$ and a common prime factor of $a+\zeta^i b$ and $a+\zeta^j b$, with $i>j$. Observe that $(a+\zeta^i b)-(a+\zeta^j b) = b\zeta^j(1-\zeta^{i-j})$ and since $\mathfrak{P}$ must divide the difference it can not divide b because in this case it would also divide a, and since a,b are integers p would divide both. So $\mathfrak{P}$ must be a factor of $\zeta^i(1-\zeta^{i-j})$ but ζ^i is a unit for all i then $\mathfrak{P}$ divides $1-\zeta^{i-j}$, that is $\mathfrak{P}=\mathfrak{P}_r$. Now for the last statement in the proposition, suppose that $r \mid a+b$. Then,

$$a+\zeta^i b = a+b-b+\zeta^i b = (a+b)+(\zeta^i-1)b,$$

and since $\nu_{\mathfrak{P}_r}(\zeta^i-1)=1$ we have $\nu_{\mathfrak{P}_r}(a+\zeta^i b)=\min\{r-1,1\}=1$ ■

Corollary 2.2 *If $(a,b)=1$, then $a+b$ and $\phi_r(a,b)$ are coprime outside r. Furthermore, if $r \mid a+b$ then $\nu_r(\phi_r(a,b))=1$.*

Proof: Let p be a prime dividing $a+b$ and $\phi_r(a,b)$ and denote by $\mathfrak{P}$ a prime in $\mathbb{Q}(\zeta)$ above p. $\mathfrak{P}$ must divide at least one of the factors $a+\zeta^i b$. Since a,b are integers $\mathfrak{P}$ can not divide b then it follows from

$$a+b = a+\zeta^i b-\zeta^i b+b = (a+\zeta^i b)+(1-\zeta^i)b$$

that $\mathfrak{P}=\mathfrak{P}_r$. Moreover, if $r \mid a+b$ it follows from the proposition that $\nu_{\mathfrak{P}_r}(a+\zeta^i b)=1$ for all i then $\nu_{\mathfrak{P}_r}(\phi_r(a,b))=r-1$ thus $\nu_r(\phi_r(a,b))=1$. ■

Proposition 2.3 *Let $(a,b)=1$ and $l \not\equiv 1$ (mod r) be a prime dividing a^r+b^r. Then $l \mid a+b$.*

Proof: Since l divides a^r+b^r, $l \nmid ab$. Let b_0 be the inverse of $-b$ modulo l. We have $a^r \equiv (-b)^r \pmod{l}$, hence $(ab_0)^r \equiv 1 \pmod{l}$. Thus the multiplicative order of ab_0 in $\mathbb{F}_l$ is 1 or r. From the congruence $ab_0 \equiv 1 \pmod{l}$ it follows $a+b \equiv 0 \pmod{l}$. If $l \nmid a+b$ then the order of ab_0 is r and $l \equiv 1 \pmod{r}$. ■

From now on we particularize to $r=13$ and we denote ϕ_{13} only by ϕ. We have $x^{13}+y^{13}=(x+y)\phi(x,y)$, where

$$\begin{aligned}\phi(x,y) &= x^{12}-x^{11}y+x^{10}y^2-x^9y^3+x^8y^4-x^7y^5+x^6y^6\\ &- x^5y^7+x^4y^8-x^3y^9+x^2y^{10}-xy^{11}+y^{12}.\end{aligned}$$

Suppose that there exists a non-trivial primitive solution (a, b, c) to (1) with $C = d\gamma$, d and γ as in theorem 1.2. Then it follows from corollary 2.2 and proposition 2.3 that there exists a non-trivial primitive solution (a, b, c_0) to

$$\phi(a, b) = c_0^p, \tag{3}$$

with $d \mid a + b$ and $13 \nmid a + b$ or to

$$\phi(a, b) = 13c_0^p \tag{4}$$

with $d \mid a + b$ and $13 \mid a + b$, where in both cases c_0 is only divisible by primes congruent to 1 modulo 13. Consider the factorization of ϕ in $\mathbb{Q}(\zeta)$

$$\phi(x, y) = \prod_{i=1}^{12} (x + \zeta^i y).$$

We have $\phi = \phi_1\phi_2$ where ϕ_i are both of degree 6 with coefficients in $\mathbb{Q}(\sqrt{13})$, given by

$$\begin{aligned} \phi_1(x, y) &= (x + \zeta y)(x + \zeta^{12} y)(x + \zeta^4 y)(x + \zeta^9 y)(x + \zeta^3 y)(x + \zeta^{10} y), \\ \phi_2(x, y) &= (x + \zeta^2 y)(x + \zeta^5 y)(x + \zeta^6 y)(x + \zeta^7 y)(x + \zeta^8 y)(x + \zeta^{11} y). \end{aligned}$$

The proof of the following corollary of proposition 2.1 is immediate.

Corollary 2.4 *Let $\mathfrak{P}_{13}$ be the prime of $\mathbb{Q}(\sqrt{13})$ above 13. If $(a, b) = 1$, then $\phi_1(a, b)$ and $\phi_2(a, b)$ are coprime outside $\mathfrak{P}_{13}$. Moreover, $\nu_{\mathfrak{P}_{13}}(\phi_i(a, b)) = 1$ or 0 for both i if $13 \mid a + b$ or $13 \nmid a + b$, respectively.*

Corollary 2.4 and the existence of a solution (a, b, c_0) to (3) or (4) implies that for some unit μ there exists a solution (a, b, c_1) (with c_1 an integer in $\mathbb{Q}(\sqrt{13})$) to the equation

$$\phi_1(a, b) = \mu c_1^p, \tag{5}$$

with $d \mid a + b$ and $13 \nmid a + b$ or to

$$\phi_1(a, b) = \mu\sqrt{13}c_1^p, \tag{6}$$

with $d \mid a + b$ and $13 \mid a + b$, respectively.

Observe that proposition 2.3 and the form of equation (1) implies that $13 \mid c$ is equivalent to $13 \mid a + b$. Moreover, proposition 2.3 also guarantees that when passing from the equation in (I) of Theorem 1.2 to equations (5) or (6) the prime factors of γ can be supposed to divide $a + b$. Since this information will not be necessary to the proof of both parts of theorem 1.2 we can assume that $\gamma = 1$. It will also be clear from the proof that the unit μ can be supposed to be 1.

3 The Frey-Hellegouarch curves

Let σ be the generator of $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ and K the subfield (of degree 6) fixed by σ^6. Consider the factorization $\phi_1 = f_1 f_2 f_3$ where

$$\begin{cases} f_1(x, y) = (x + \zeta y)(x + \zeta^{12} y) = x^2 + (\zeta + \zeta^{12})xy + y^2 \\ f_2(x, y) = (x + \zeta^4 y)(x + \zeta^9 y) = x^2 + (\zeta^4 + \zeta^9)xy + y^2 \\ f_3(x, y) = (x + \zeta^3 y)(x + \zeta^{10} y) = x^2 + (\zeta^3 + \zeta^{10})xy + y^2 \end{cases}$$

are the degree two factors of ϕ_1 with coefficients in K. Now we are interested in finding a triple (α, β, γ) such that

$$\alpha f_1 + \beta f_2 + \gamma f3 = 0.$$

Solving a linear system in the coefficients of the f_i we find that one of its infinite solutions in $\mathcal{O}_K^3$ is given by

$$\begin{cases} \alpha = -\zeta^{10} + \zeta^9 + \zeta^4 - \zeta^3 \\ \beta = \zeta^{12} - \zeta^9 - \zeta^4 + \zeta \\ \gamma = -\zeta^{12} + \zeta^{10} + \zeta^3 - \zeta \end{cases}$$

and verifies $\nu_{\mathfrak{P}_{13}}(\alpha) = \nu_{\mathfrak{P}_{13}}(\beta) = \nu_{\mathfrak{P}_{13}}(\gamma) = 1$.

Suppose now that $(a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \mathcal{O}_{\mathbb{Q}(\sqrt{13})}$ is a non-trivial primitive solution to equation (5) or (6) and let $A(a, b) = \alpha f_1(a, b)$, $B(a, b) = \beta f_2(a, b)$ and $C(a, b) = \gamma f_3(a, b)$. Since

$$A + B + C = 0$$

then we can consider the Frey-curve over K with the classic form

$$E(a, b) : y^2 = x(x - A(a, b))(x + B(a, b)).$$

In the rest of this section we will denote $E(a, b)$ only by E every time it causes no ambiguity. Let $\mathfrak{P}_2$ denote the prime of K above 2 (is inert). To the curves $E(a, b)$ are associated the following quantities:

$$\begin{aligned} \Delta(E) &= 2^4(ABC)^2 = 2^4(\alpha\beta\gamma)^2\phi_1(a, b)^2 \\ c_4(E) &= 2^4(A^2 + AB + B^2) = 2^4(AB + BC + AC) \\ c_6(E) &= -2^5(C + 2B)(A + 2B)(2A + B) \\ j(E) &= 2^8\frac{(A^2 + AB + B^2)^3}{(ABC)^2} \end{aligned}$$

and since $(\alpha\beta\gamma) = \mathfrak{P}_{13}^3$ the discriminant of E takes the following values

$$\Delta(E) = \begin{cases} \mathfrak{P}_2^4\mathfrak{P}_{13}^6 c^{2p} \text{ if } 13 \nmid a + b \\ \mathfrak{P}_2^4\mathfrak{P}_{13}^{12} c^{2p} \text{ if } 13 \mid a + b \end{cases}$$

Proposition 3.1 *Let $\mathfrak{P}$ be a prime of K distinct from $\mathfrak{P}_2$ and $\mathfrak{P}_{13}$. The curves $E(a, b)$ have good or multiplicative reduction at $\mathfrak{P}$. Moreover, the curves have good ($\nu_{\mathfrak{P}_{13}}(N_E) = 0$) or bad additive ($\nu_{\mathfrak{P}_{13}}(N_E) = 2$) reduction at $\mathfrak{P}_{13}$ if $13 \mid a + b$ or $13 \nmid a + b$, respectively. In particular, E has multiplicative reduction at primes dividing c.*

Proof: To the results used in this proof we followed [22]. Let $\mathfrak{P}$ be has in the hypothesis and observe that $\upsilon_{\mathfrak{P}}(\Delta(E)) = 2p\upsilon_{\mathfrak{P}}(c)$. Then if $\mathfrak{P} \nmid c$ we have $\upsilon_{\mathfrak{P}}(\Delta) = 0$ and the curve has good reduction. It follows from proposition 2.1 that $A(a, b)$, $B(a, b)$ and $C(a, b)$ are pairwise coprime outside $\mathfrak{P}_{13}$ and recall that the three are divisible by $\mathfrak{P}_{13}$. If $\mathfrak{P} \mid c$ then $\mathfrak{P}$ must divide only one among A, B or C. From the form of c_4 it can be seen that $\upsilon_{\mathfrak{P}}(c_4) = 0$. Also, $\upsilon_{\mathfrak{P}}(\Delta) > 0$ thus E has multiplicative reduction at $\mathfrak{P}$. Moreover, we see from $\Delta(E)$ that if $13 \mid a + b$ the equation is not minimal and E has good reduction at $\mathfrak{P}_{13}$. On the other hand if $13 \nmid a + b$ then $\nu_{\mathfrak{P}_{13}}(\Delta) = 6$ and $\nu_{\mathfrak{P}_{13}}(c_4) > 0$ hence bad additive reduction at $\mathfrak{P}_{13}$.

$\blacksquare$

Proposition 3.2 *The short Weierstrass model of the curves $E(a, b)$ is defined over $\mathbb{Q}(\sqrt{13})$.*

Proof: First observe that $\sigma^4 \pmod{\sigma^6}$ generates $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{13}))$. Since the curves E are defined over K they are invariant under σ^6 and in particular $j(E)$ is invariant by σ^6 by definition. We also have that

$$\sigma^4(A) = B, \qquad \sigma^4(B) = C, \qquad \sigma^4(C) = A,$$

and from

$$j(E) = 2^8 \frac{(AB + BC + CA)^3}{(ABC)^2}$$

it is clear that j is also invariant under σ^4. Then the j-invariant that *a priori* belonged to K of degree 6, in reality is in $\mathbb{Q}(\sqrt{13})$. Now we write $E(a, b)$ in the short Weierstrass form to get a model

$$\begin{cases} E_0 : y^2 = x^3 + a_4 x + a_6, \text{ where} \\ a_4 = -432(AB + BC + CA) \\ a_6 = -1728(2A^3 + 3A^2B - 3AB^2 - 2B^3) \end{cases}$$

Since a_4 is clearly invariant under σ^4 and

$$\begin{aligned} a_6 &= -1728(2A^3 + 3A^2B - 3AB^2 - 2B^3) = \\ &= -1728(2(-B - C)^3 + 3(-B - C)^2B - 3(-B - C)B^2 - 2B^3) = \\ &= -1728(2B^3 + 3B^2C - 3BC^2 - 2C^3) = \sigma^4(a_6) \end{aligned}$$

we conclude that the short Weierstrass model is already defined over $\mathbb{Q}(\sqrt{13})$ $\blacksquare$

Writing E in the short Weierstrass form we get an elliptic curve E_0 defined over $\mathbb{Q}(\sqrt{13})$ given by

$$E_0 : y^2 = x^3 + a_4(a, b)x + a_6(a, b),$$

$$\begin{aligned} a_4(a, b) &= (216w - 2808)a^4 + (-1728w + 5616)a^3b \\ &\quad +(1728w - 11232)a^2b^2 + (-1728w + 5616)ab^3 \\ &\quad +(216w - 2808)b^4, \\ a_6(a, b) &= (-8640w + 44928)a^6 + (49248w - 235872)a^5b \\ &\quad +(-129600w + 471744)a^4b^2 + (152928w - 662688)a^3b^3 + \\ &\quad +(-129600w + 471744)a^2b^4 + (49248w - 235872)ab^5 + \\ &\quad +(-8640w + 44928)b^6 + (50193w + 182520)b^6, \end{aligned}$$

where $w^2 = 13$. Writing a curve in short Weierstrass form changes the values of Δ, c_4 and c_6 according to $\Delta(E_0) = 6^{12}\Delta(E)$, $c_4(E_0) = 6^4c_4(E)$ and $c_6(E_0) = 6^6c_6(E)$. Observe that 2 is inert in $\mathbb{Q}(\sqrt{13})$ (and in K) and denote by 2 and w the ideals in $\mathbb{Q}(\sqrt{13})$ above 2 and 13, respectively.

Proposition 3.3 *The possible values for the conductors of* $E_0(a,b)$ *are*

$$N_{E_0} = 2^s w^2 rad(c),$$

where $s = 3, 4$ *and* $rad(c)$ *is the product of the prime factors of* c*. Moreover, if* $2 \mid a+b$ *then* $s = 3$ *if* $4 \mid a+b$ *and* $s = 4$ *if* $4 \nmid a+b$*.*

Proof: As in proposition 3.1 we followed the results in [22] to compute the conductor. Since the primes dividing 6 do not ramify in $K/\mathbb{Q}(\sqrt{13})$ and do not divide c the conductor of E_0 and E is same at these primes.

Since $(w) = \mathfrak{P}_{13}^3$ in K we see from Proposition 3.1 that $\nu_w(\Delta(E_0)) = 4$ or 2 if $13 \mid a+b$ or $13 \nmid a+b$, respectively. Also, $\nu_w(c_4(E_0)) > 0$ and since we are in characteristic ≥ 5 this implies that the equation is minimal and has bad additive reduction with $\nu_w(N_{E_0}) = 2$.

It easily can be seen that $\nu_2(\Delta(E_0)) = 16$, $\nu_2(c_6(E_0)) = 11$ and $\nu_2(c_4(E_0)) \geq 8$. Table IV in [22] tell us that the equation is not minimal and after a change of variables we have $\nu_2(\Delta(E_0)) = 4$, $\nu_2(c_6(E_0)) = 5$ and $\nu_2(c_4(E_0)) \geq 4$. We check that in the columns where $\nu_2(c_6(E_0)) = 5$ and $\nu_2(\Delta(E_0)) = 4$ we have $\nu_2(N_{E_0}) = 2, 3, 4$. The conductors at 2 of the curves $E(1,-1)$ and $E(1,1)$ are 2^3 and 2^4, respectively. The case $s = 2$ never happens. This is a direct consequence of Proposition 2 in [22] by taking $r = w/2 + 1/2$ if $(a,b) \equiv (1,0), (0,1) \pmod 2$ or $r = 0$ if $(a,b) \equiv (1,1) \pmod 2$. Since the same proposition 2 guarantees that all the possible conductors at 2 will occur for the pairs (a,b) modulo 4 we use SAGE to compute the conductor for all these pairs and easily verify the statement by inspection.

■

From now on we will write E to denote E_0.

4 The Galois representations of $E(a,b)$

Let $\rho_{E,p} : G_{\mathbb{Q}(\sqrt{13})} \to GL_2(\mathbb{Q}_p)$ be the p-adic representation associated with E and $\bar{\rho}_{E,p}$ its reduction modulo p. In this section we will prove that $\rho_{E,p}$ is modular and that $\bar{\rho}_{E,p}$ is irreducible for $p > 97$. These results together allow us to apply the lowering the level theorems for Hilbert modular forms.

4.1 Modularity

Theorem 4.1 *Let* F *be a totally real cyclic number field and* E *and elliptic curve defined over* F*. Suppose that 3 splits in* F *and* E *has good reduction at the primes above 3. Then* E *is modular.*

Let N_E denote the conductor of E and put $\rho = \rho_{E,3}$. Since the representation $\bar{\rho} = \bar{\rho}_{E,3} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{13})) \to \mathrm{GL}_2(\mathbb{F}_3)$ is odd it is absolutely irreducible if and only if it is irreducible. The following key lemma is a known result and a consequence of the work of Langlands-Tunnell. For references where it is used see [21], [26], [14] and [17].

Lemma 4.2 *If* $\bar{\rho}$ *is irreducible then it is modular arising from an Hilbert newform over* F *of parallel weight* $(2,2)$*. Moreover, if* ρ *is ordinary we can suppose that* f *is ordinary at 3.*

Proof: Let t be a prime in F above 3.

For the second statement of the theorem, by hypothesis we know that

$$\bar{\rho}|I_t = \begin{pmatrix} \bar{\chi} & * \\ 0 & 1 \end{pmatrix},$$

where χ is the 3-adic cyclotomic character. It is know from the work of Jarvis, Rajaei and Fujiwara that since $\bar{\rho}_{f,3}|D_t \equiv \bar{\rho}|D_t$ and $\rho|D_t$ is a Barsotti-Tate representation for all t (because E has good reduction at 3) we can choose f to have parallel weight $(2,2)$ and level coprime with 3, hence $\rho_{f,3}|D_t$ is also Barsotti-Tate. Since $\bar{\rho}_{f,3}|D_t$ is ordinary we can apply a result of Breuil (see [4], chapter 9) to conclude that $\rho_{f,3}$ is ordinary. ∎

We will now prove the theorem. A similar argument but over $\mathbb{Q}$ was given by the first author in [11].

Proof (of theorem 4.1): We divide the proof into three cases:

(1) Suppose that $\bar{\rho}$ and $\bar{\rho}|G_{\mathbb{Q}(\sqrt{-3})}$ are both abs. irreducible. Here we apply corollary 2.1.3 in [19]. Condition (1) holds because E has good reduction at the primes above 3 and 3 splits in F. Lemma 4.2 guarantees condition (2) and (3) is obvious. Then ρ is modular.

(2) Suppose that $\bar{\rho}$ is abs. irreducible and $\bar{\rho}|G_{F(\sqrt{-3})}$ abs. reducible. This means that the image of $\mathbb{P}(\bar{\rho})$ is Dihedral. Namely, that the image of $\bar{\rho}$ is contained in the normalizer N of a Cartan subgroup C of $GL_2(\bar{\mathbb{F}}_3)$ but not contained in C. Moreover, the restriction to $\mathbb{Q}(\sqrt{-3})$ of our representation has its image inside C. Thus, the composition of $\bar{\rho}$ with the quotient N/C,

$$\mathrm{Gal}(\bar{\mathbb{Q}}/F) \to N \to N/C, \tag{7}$$

gives the quadratic character of $F(\sqrt{-3})/F$ which ramifies at 3 because 3 is unramified in F.

Let t be a prime in F above 3. Since E has good reduction at 3, the restriction of the residual representation $\bar{\rho}$ to the inertia subgroup I_t has only two possibilities

$$\bar{\rho}|I_t = \begin{pmatrix} \bar{\chi} & * \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi_2 & 0 \\ 0 & \psi_2^3 \end{pmatrix},$$

where χ is the 3-adic cyclotomic character and ψ_2 is the fundamental character of level 2.

If we suppose that $\bar{\rho}|I_t$ acts through level 2 fundamental characters, the image of I_t by $\mathbb{P}(\bar{\rho})$ gives a cyclic group of order $4 > 2$, thus it has to be contained in $\mathbb{P}(C)$ (if it not contained in $\mathbb{P}(C)$ and has order 4 it must be isomorphic to $C_2 \times C_2$). But this implies that the quadratic character defined by composition (7) should be unramified at 3, contradicting the fact that this character corresponds to $F(\sqrt{-3})$. Thus we can assume that we are in the first case, that is, $\bar{\rho}|I_t$ is reducible. Since ρ is crystalline with Hodge-Tate weights 0 and 1 and 3 splits in F we apply a result of Breuil (see [4], chapter 9) to conclude that $\rho|D_t$ is reducible hence ordinary. Indeed, we can apply this result of classification of crystalline

representations at 3 because the highest Hodge-Tate weight is $w = 1$ and so $3 > w + 1$. Since t is arbitrary the previous holds for all primes t above 3 hence ρ is ordinary. Therefore, we can suppose that the form f given by Lemma 4.2 is ordinary thus all conditions are satisfied to apply Theorem 5.1 in [25] to conclude that ρ is modular.

(3) Suppose that $\bar{\rho}$ is abs. reducible. We exclude again the case of the fundamental characters of level 2, but this time this is automatic because of reducibility. Then

$$\bar{\rho}^{ss} = \epsilon \oplus \epsilon^{-1}\bar{\chi},$$

where ϵ ramifies only at the primes of N_E. Also, since the representation is odd and F is real, reducibility must take place over $\mathbb{F}_3$. Again by the result of Breuil on crystalline representations we conclude that $\rho|D_t$ is reducible. Observe now that ϵ must be quadratic because $\mathbb{F}_3^*$ has two elements then $\epsilon/(\epsilon^{-1}\bar{\chi}) = \bar{\chi}$ and the extension $F(\sqrt{-3})/\mathbb{Q}$ is abelian because F is cyclic and disjoint from $\mathbb{Q}(\sqrt{-3})$. This establishes all the conditions of theorem A in [24] thus ρ is modular.

■

4.2 Irreducibility

Proposition 4.3 *Let $p > 97$ be a prime. The representation $\bar{\rho}_{E,p}$ is absolutely irreducible.*

Proof: Since $\bar{\rho}_{E,p}$ is odd and $\mathbb{Q}(\sqrt{13})$ is totally real it is known that $\bar{\rho}_{E,p}$ is absolutely irreducible if and only if it is irreducible then we only need to rule out the case where $\bar{\rho}_{E,p}$ is reducible and has the form

$$\bar{\rho}_{E,p} = \begin{pmatrix} \epsilon^{-1}\chi_p & * \\ 0 & \epsilon \end{pmatrix}, \tag{8}$$

where χ_p is the mod p cyclotomic character and ϵ is a character of $G_{\mathbb{Q}(\sqrt{13})}$ with values in $\mathbb{F}_p$. Since the image of inertia at semistable primes is of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ the conductor of ϵ only contains additive primes. By the work of Carayol the conductor at bad additive primes of $\bar{\rho}_{E,p}$ is the same as that of $\rho_{E,p}$. Since the conductors of ϵ and ϵ^{-1} are equal it follows from proposition 3.3 that the cond$(\epsilon) = \mathfrak{P}_2\mathfrak{P}_{13}$ or $\mathfrak{P}_2^2\mathfrak{P}_{13}$. The characters of $G_{\mathbb{Q}(\sqrt{13})}$ with conductor dividing $\mathfrak{P}_2^2\mathfrak{P}_{13}$ are in correspondence with the characters of the finite group

$$H = (\mathcal{O}_{\mathbb{Q}(\sqrt{13})}/\mathfrak{P}_2^2\mathfrak{P}_{13})^* \cong \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The group of characters of H is dual of H then all the characters have order dividing 12. In particular ϵ is a root of the polynomial $q_1 := x^{12} - 1 \pmod{p}$ for any p. Let $\mathfrak{P}_3$ be a prime above 3. By taking traces on equality (8) we get

$$a_{\mathfrak{P}_3} \equiv \epsilon(\mathrm{Frob}_{\mathfrak{P}_3}) + 3\epsilon^{-1}(\mathrm{Frob}_{\mathfrak{P}_3}) \qquad \text{(mod p)},$$

which implies that $\epsilon(\mathrm{Frob}_{\mathfrak{P}_3})$ satisfies for any p the polynomial $q_2 := x^2 - a_{\mathfrak{P}_3}x + 3 \pmod{p}$. Let $\zeta = \zeta_{12}$, then the resultant of q_1 and q_2 is given by

$$\begin{aligned} \mathrm{res}(q_1, q_2) &= \prod_{i=1}^{12} \left(\frac{a_{\mathfrak{P}_3} + \sqrt{a_{\mathfrak{P}_3}^2 - 12}}{2} - \zeta^i\right)\left(\frac{a_{\mathfrak{P}_3} - \sqrt{a_{\mathfrak{P}_3}^2 - 12}}{2} - \zeta^i\right) \\ &= \prod_{i=1}^{12} (\zeta^{2i} - a_{\mathfrak{P}_3}\zeta^i + 3) \end{aligned}$$

Since $|a_{\mathfrak{P}_3}| \leq 3$ we compute all the possibilities to the product above to find that the greater prime divisor appearing as a possible factor is 97. Therefore, since $\epsilon(\mathrm{Frob}_{\mathfrak{P}_3})$ is a common root of the $q_i \pmod{p}$ then $\mathrm{res}(q_1, q_2) \equiv 0 \pmod{p}$, which is impossible if $p > 97$. Thus $\bar{\rho}_{E,p}$ is absolutely irreducible if $p > 97$ ■

With the theorems above we are now able to lower the level. See Jarvis-Meekin [18] for an application of the level lowering results for Hilbert modular forms in [23], [15] and [16]. In the present case we apply these results along the same lines. Denote by $S_2(N)$ the set of Hilbert modular cusp forms of parallel weight $(2, 2)$ and level N. It follows from the modularity that there exists a newform f_0 in $S_2(\mathfrak{P}_2^i\mathfrak{P}_{13}^2\mathrm{rad}(c))$ where $i = 3$ or 4 such that $\rho_{E,p}$ is isomorphic to the p-adic representation associated with f_0, which we denote by $\rho_{f_0,p}$. Since the semistable primes of E, i.e. those dividing c, appear to a p-th power in the discriminant $\Delta(E)$ we know by an argument of Hellegouarch that the representation $\bar{\rho}_{E,p}$ will not ramify at these primes. Furthermore, when reducing to the residual representation the conductor at the bad additive primes can not decrease hence $\bar{\rho}_{E,p}$ has conductor equal to that of $\rho_{E,p}$ without the factor $\mathrm{rad}(c)$, that is $\mathfrak{P}_2^i\mathfrak{P}_{13}^2$ with $i = 3$ or 4. Since $\rho_{E,p}$ is modular then $\bar{\rho}_{E,p}$, when irreducible, is modular and by the results on level lowering for Hilbert modular forms we know that there exists a newform f in $S_2(\mathfrak{P}_2^i\mathfrak{P}_{13}^2)$ such that its associated mod p Galois representation satisfies

$$\rho_{E,p} \equiv \rho_{f_0,p} \equiv \rho_{f,p} \pmod{\mathfrak{P}}. \tag{9}$$

5 Eliminating Newforms

In this section we will find a contradiction to congruence (9). This shows that the Frey-curves associated with primitive non-trivial first case solutions (a, b, c) to equation (5) or (6) can not exist and ends the proof of part (I) in Theorem 1.2. To find the desired contradiction we use the trace values $a_L(\rho_{E,p})$ and $a_L(\rho_{f,p})$ for some primes L of $\mathbb{Q}(\sqrt{13})$ and the Hilbert modular newforms f in the spaces predicted in the previous section. Let $w \in \mathbb{Q}(\sqrt{13})$ be such that $w^2 = 13$ and consider the following prime ideals in $\mathbb{Q}(\sqrt{13})$:

$$\begin{cases} L_2 = \langle 2 \rangle, \quad L_{13} = \langle w \rangle \\ L_3^0 = \langle \frac{1}{2}(w+1) \rangle, \quad L_3^1 = \langle \frac{1}{2}(-w+1) \rangle, \\ L_{17}^0 = \langle \frac{1}{2}(w+9) \rangle, \quad L_{17}^1 = \langle \frac{1}{2}(-w+9) \rangle, \\ L_{23}^0 = \langle \frac{1}{2}(-3w-5) \rangle, \quad L_{23}^1 = \langle \frac{1}{2}(-3w+5) \rangle, \\ L_{29}^0 = \langle \frac{1}{2}(3w+1) \rangle, \quad L_{29}^1 = \langle \frac{1}{2}(3w-1) \rangle, \\ L_5 = \langle 5 \rangle, \quad L_7 = \langle 7 \rangle, \quad L_{11} = \langle 11 \rangle. \end{cases}$$

On one hand, to obtain the values of $a_L(\rho_{f,p})$, with the aid of John Voight we used algorithms to compute Hilbert modular forms implemented in MAGMA [3] (an expository account can be found in [10]). John Voight gave us two lists corresponding to all forms with integer coefficients such that $a_{L_2} = a_{L_{13}} = 0$ and of levels $\mathfrak{P}_2^s\mathfrak{P}_{13}^2$ for $s = 3, 4$. With MAGMA we have done the same to all dividing levels and by putting together both informations we obtained all newforms in the spaces $S_2(\mathfrak{P}_2^s\mathfrak{P}_{13}^2)$ for $s = 3, 4$ such that $\mathbb{Q}_f = \mathbb{Q}$. A list of coefficients corresponding to the newforms obtained this way can be found in the appendix A. Moreover, a consequence of the method used is that any newform in the two previous spaces with $\mathbb{Q}_f$ strictly containing $\mathbb{Q}$ must have a Fourier coefficient outside $\mathbb{Q}$ at the prime L_3^0 above 3. John Voight also computed the factorization of the characteristic polynomial of the Hecke operator $T_{L_3^0}$ in both spaces (see appendix B).

On the other hand, for every prime L in $\mathbb{Q}(\sqrt{13})$ of good reduction for E, such that L is above a rational prime $l \leq 29$ and $l \neq 19$, we use SAGE to go through all the possible residual elliptic curves for all pairs $(a, b) \in \mathbb{F}_l \times \mathbb{F}_l$ and compute all the possible values for $a_L(\rho_{E,p}) = a_L(E)$:

$$\begin{cases} a_{L_3^0} \in \{-3, -1\}, \\ a_{L_3^1} \in \{-3, -1, 1\}, \\ a_{L_5} \in \{-6, -2, 2\}, \\ a_{L_7} \in \{11, -11, -1, -5\}, \\ a_{L_{11}} \in \{-15, 3, 5, -7, 9, -1, 15\}, \\ a_{L_{17}^0} \in \{1, 3, 5, 7, -3, -1\}, \\ a_{L_{17}^1} \in \{3, 5, 7, -7, -5, -3\}, \\ a_{L_{23}^0} \in \{1, 3, 5, 7, -9, -7, -5, -3\}, \\ a_{L_{23}^1} \in \{1, 3, 7, -9, -3, -1\}, \\ a_{L_{29}^0} \in \{1, 3, 5, -9, -7, -5, -3, -1\}, \\ a_{L_{29}^1} \in \{1, 3, 5, 9, -9, -7, -5, -3, -1\} \end{cases}$$

Before proceeding to eliminate the newforms we divide them into two sets:

- S1: The newforms in $S_2(\mathfrak{P}_2^i\mathfrak{P}_{13}^2)$ for $i = 3, 4$ such that $\mathbb{Q}_f = \mathbb{Q}$.
- S2: The newforms in the same levels with $\mathbb{Q}_f$ strictly containing $\mathbb{Q}$.

Note that equations (5) and (6) have trivial solutions $(1, 1, 1)$, $\pm(0, 1, 1)$, $\pm(1, 0, 1)$ and $(1, -1, 1)$, $(-1, 1, 1)$, respectively. These solutions correspond to the Frey-curves $E(1, 1)$, $E(0, 1)$ and $E(1, -1)$ that indeed exist and so there must be newforms associated with them in S1 which a priori will not be possible to eliminate only by comparing the a_L.

Going through all the forms in $S1$ and comparing the corresponding values of the $a_L's$ with the possibilities for our Frey-curves we immediately eliminate all except 4 newforms. Here we have eliminated a newform if one of its coefficients a_L is not on the corresponding list above. This can be done because the value of p in the statement of Theorem 1.2 is very large hence congruence (9), when specified at a trace at L for a prime L of small norm, does not hold modulo such large prime p unless $a_L(f) = a_L(E)$. For example, the first form in the

appendix satisfies $a_{L_5}(f) = -9$ and since $a_{L_5}(E) \in \{-6, -2, 2\}$ it is clear that $-9 \equiv -6, -2, 2 \pmod{p}$ can not hold for $p > 11$. The four remaining newforms correspond to the trivial solutions above plus the twist by -1 of $E(1,1)$. The one associated with $E(1,1)$ has level $\mathfrak{P}_2^4 \mathfrak{P}_{13}^2$ and the other three $\mathfrak{P}_2^3 \mathfrak{P}_{13}^2$. In table 1 we list their first eigenvalues.

	$a_{L_3^0}$	$a_{L_3^1}$	$a_{L_{17}^0}$	$a_{L_{17}^1}$	$a_{L_{23}^0}$	$a_{L_{23}^1}$	a_{L_5}	$a_{L_{29}^0}$	$a_{L_{29}^1}$	a_{L_7}	$a_{L_{11}}$
f_1	-1	1	7	3	1	7	2	-7	-3	-1	3
f_2	-1	1	3	7	-7	-1	2	-3	-7	-1	3
f_3	-1	-3	-1	-5	5	-9	-6	-3	1	-5	15
f_4	-3	-1	1	-3	-3	-9	-2	-7	5	-11	-15

Table 1: Values of a_L

To be able to eliminate these newforms we need to use the extra conditions on d and $a+b$. Recall that the solutions (a,b,c) to equation (5) or (6) satisfy $d \mid a+b$. Recomputing the possibilities for some a_L but with this extra condition we find that $a_{L_3^0} = -3$ and $a_{L_3^1} = -1$ (if $d=3$) , $a_{L_5} = -2$ (if $d = 5$) , $a_{L_7} = -11$ (if $d = 7$) or $a_{L_{11}} = -15$ (if $d = 11$). By checking in table 1 we see that any of the previous conditions is enough to eliminate all f_i except for f_4. Actually, f_4 is the newform associated with the trivial solution $(1,-1,0)$ and can not be eliminated this way as expected. Finally, if we assume that the solution is first case, Proposition 3.1 together with condition $13 \nmid a+b$ guarantees that when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ the representations $\rho_{f_4,p}$ and $\rho_{E,p}$ will have different inertia at $\mathfrak{P}_{13}$ and thus can not be isomorphic modulo $\mathfrak{P}$.

To finish the argument we have to eliminate also the newforms in $S2$. Recall that we know the factorization (appendix B) of the characteristic polynomial of $T_{L_3^0}$ which we denote by p_3. If for f in $S2$ congruence (9) holds we also have

$$a_{L_3^0}(E) \equiv c_{L_3^0}(f) \pmod{\mathfrak{P}}.$$

Let $p_c(x)$ be the minimal polynomial of $c_{L_3^0}(f)$ which must be a non-linear factor of p_3. Thus,

$$p_c(a_{L_3^0}(E)) \equiv p_c(c_{L_3^0}(f)) \equiv 0 \pmod{\mathfrak{P}} \tag{10}$$

and $p_c(a_{L_3^0}(E)) \neq 0$ because $c_{L_3^0}(f) \notin \mathbb{Z}$. Since $a_{L_3^0}(E) \in \{3, -1\}$ by computing $p_c(3)$ and $p_c(-1)$ for all p_c a non-linear factors of p_3 we have all the possibilities for $p_c(a_{L_3^0}(E))$ and we can see that congruence (10) can not hold if $p > 4992539$. Therefore (9) also can not hold if $p > 4992539$ and this ends the proof of part (I) in Theorem 1.2. ∎

Remark 5.1 *It is also possible to eliminate the newforms in $S2$ without knowing the factorization of p_3 but this would result in the bound $p > 2^{14546}$ for the exponent. Indeed, let $p_c = \sum r_n x^n$ be the minimal polynomial of a non integer $c_{L_3}(f)$. All the roots $c_{L_3}^{\sigma}$ satisfy the Weil bound since they are coefficients of the conjugated form f^{σ}. Moreover, by knowing the dimension of $S_2(\mathfrak{P}_2^s \mathfrak{P}_{13}^2)$ we can bound all $|r_n|$ using the binomial coefficients. Putting these bounds together we find only a finite number of possibilities for the non-zero value $p_c(a_{L_3}(E))$. The details for this argument can be found in the first version of this work at http://arxiv.org/abs/1112.4521*

Part (II) now follows easily from the proof of part (I). First note that as before it follows from the factorization

$$x^{26} + y^{26} = (x^2 + y^2)\phi(x^2, y^2) = 10z^p, \tag{11}$$

Proposition 2.3 and Corollary 2.2, that a solution (a, b, c) must verify $10 \mid a^2+b^2$. To a primitive solution (a, b, c) we now attach the Frey-curve $E(a^2, b^2)$. Observe also that $4 \nmid a^2 + b^2$ by looking modulo 4. It now follows from Proposition 3.3, modularity and lowering the level that the set S1 will only have newforms of level $\mathfrak{P}_2^4\mathfrak{P}_{13}^2$. This means that after comparing the values a_L as before, we eliminate all newforms except for the one corresponding to the curve $E(1, 1)$. As we already know, the extra restriction $5 \mid a^2 + b^2$ is enough to deal with this newform. In fact, recall that in this case the Frey curve has $a_5 = -2$, and this is different from the corresponding coefficient a_5 of $E(1, 1)$. The newforms in S2 can be eliminated exactly as in the proof of part (I). ∎

6 Appendix

6.1 A: Tables with values a_L

$a_{L_3^0}$	$a_{L_3^1}$	$a_{L_{17}^0}$	$a_{L_{17}^1}$	$a_{L_{23}^0}$	$a_{L_{23}^1}$	a_{L_5}	$a_{L_{29}^0}$	$a_{L_{29}^1}$	a_{L_7}	$a_{L_{11}}$
-3	-3	-3	-3	-4	-4	-9	2	2	-13	-18
-3	-1	-5	-1	-9	5	-6	1	-3	-5	15
-3	-1	-1	3	3	9	2	-7	5	11	15
-3	-1	1	-3	-3	-9	-2	-7	5	-11	-15
-3	-1	5	1	9	-5	6	1	-3	5	-15
-3	1	-7	-7	3	-1	6	7	-9	1	9
-3	1	7	7	-3	1	-6	7	-9	-1	-9
-1	-3	-3	1	-9	-3	-2	5	-7	-11	-15
-1	-3	-1	-5	5	-9	-6	-3	1	-5	15
-1	-3	1	5	-5	9	6	-3	1	5	-15
-1	-3	3	-1	9	3	2	5	-7	11	15
-1	-1	3	3	-6	-6	1	0	0	5	22
-1	1	-7	-3	-1	-7	-2	-7	-3	1	-3
-1	1	-5	7	5	3	-6	1	5	-1	3
-1	1	5	-7	-5	-3	6	1	5	1	-3
-1	1	7	3	1	7	2	-7	-3	-1	3
-1	3	-1	7	2	2	-7	-8	0	1	6
-1	3	1	-7	-2	-2	7	-8	0	-1	-6
0	0	6	6	8	8	-6	2	2	-10	-18
1	-3	-7	-7	-1	3	6	-9	7	1	9
1	-3	7	7	1	-3	-6	-9	7	-1	-9
1	-1	-7	5	-3	-5	6	5	1	1	-3
1	-1	-3	-7	-7	-1	-2	-3	-7	1	-3
1	-1	3	7	7	1	2	-3	-7	-1	3
1	-1	7	-5	3	5	-6	5	1	-1	3
1	1	-3	-3	-1	-1	6	3	3	13	21
1	1	-3	-3	0	0	-1	6	6	-13	14
1	1	-3	-3	4	4	-9	-6	-6	11	-18
1	1	3	3	-4	-4	9	-6	-6	-11	18

Table 2: a_L values for newforms of level $\mathfrak{P}_2^3\mathfrak{P}_{13}^2$

$a_{L_3^0}$	$a_{L_3^1}$	$a_{L_{17}^0}$	$a_{L_{17}^1}$	$a_{L_{23}^0}$	$a_{L_{23}^1}$	a_{L_5}	$a_{L_{29}^0}$	$a_{L_{29}^1}$	a_{L_7}	$a_{L_{11}}$
1	1	3	3	1	1	-6	3	3	-13	-21
3	-1	-7	1	-2	-2	7	0	-8	-1	-6
3	-1	7	-1	2	2	-7	0	-8	1	6
-3	-3	-3	-3	-1	-1	6	-1	-1	13	-3
-3	-3	3	3	1	1	-6	-1	-1	-13	3
-3	-1	-7	-7	-3	-1	-6	-7	9	1	9
-3	-1	-7	1	-2	2	-7	0	8	-1	-6
-3	-1	7	-1	2	-2	7	0	8	1	6
-3	-1	7	7	3	1	6	-7	9	-1	-9
-3	1	-7	1	2	2	7	0	-8	-1	-6
-3	1	-5	-1	9	5	6	-1	3	-5	15
-3	1	-1	3	-3	9	-2	7	-5	11	15
-3	1	1	-3	3	-9	2	7	-5	-11	-15
-3	1	5	1	-9	-5	-6	-1	3	5	-15
-3	1	7	-1	-2	-2	-7	0	-8	1	6
-3	3	-3	-3	1	-1	-6	1	1	13	-3
-3	3	-3	-3	4	-4	9	-2	-2	-13	-18
-3	3	3	3	-4	4	-9	-2	-2	13	18
-3	3	3	3	-1	1	6	1	1	-13	3
-2	-2	-3	-3	6	6	7	3	3	14	22
-2	-2	3	3	-6	-6	-7	3	3	-14	-22
-2	2	-3	-3	-6	6	-7	-3	-3	14	22
-2	2	-3	-3	-6	6	-1	-9	-9	-2	22
-2	2	3	3	6	-6	1	-9	-9	2	-22
-2	2	3	3	6	-6	7	-3	-3	-14	-22
-1	-3	-7	-7	-1	-3	-6	9	-7	1	9
-1	-3	-1	7	-2	2	7	8	0	1	6
-1	-3	1	-7	2	-2	-7	8	0	-1	-6
-1	-3	7	7	1	3	6	9	-7	-1	-9
-1	-1	-7	-3	1	-7	2	7	3	1	-3
-1	-1	-7	5	-3	5	-6	-5	-1	1	-3
-1	-1	-5	7	-5	3	6	-1	-5	-1	3
-1	-1	-3	-7	-7	1	2	3	7	1	-3

Table 3: a_L values for newforms of level $\mathfrak{P}_2^4\mathfrak{P}_{13}^2$

$a_{L_3^0}$	$a_{L_3^1}$	$a_{L_{17}^0}$	$a_{L_{17}^1}$	$a_{L_{23}^0}$	$a_{L_{23}^1}$	a_{L_5}	$a_{L_{29}^0}$	$a_{L_{29}^1}$	a_{L_7}	$a_{L_{11}}$
-1	-1	-3	-3	1	1	6	3	3	13	21
-1	-1	3	3	-1	-1	-6	3	3	-13	-21
-1	-1	3	3	0	0	1	6	6	13	-14
-1	-1	3	3	4	4	9	-6	-6	-11	18
-1	-1	3	7	7	-1	-2	3	7	-1	3
-1	-1	5	-7	5	-3	-6	-1	-5	1	-3
-1	-1	7	-5	3	-5	6	-5	-1	-1	3
-1	-1	7	3	-1	7	-2	7	3	-1	3
-1	1	-7	5	3	5	6	5	1	1	-3
-1	1	-3	-7	7	1	-2	-3	-7	1	-3
-1	1	-3	-3	-6	6	1	0	0	-5	-22
-1	1	-3	-3	-1	1	-6	-3	-3	13	21
-1	1	-3	-3	0	0	1	-6	-6	-13	14
-1	1	-3	-3	3	-3	10	9	9	-11	5
-1	1	-3	-3	4	-4	9	6	6	11	-18
-1	1	3	3	-4	4	-9	6	6	-11	18
-1	1	3	3	-3	3	-10	9	9	11	-5
-1	1	3	3	0	0	-1	-6	-6	13	-14
-1	1	3	3	1	-1	6	-3	-3	-13	-21
-1	1	3	3	6	-6	-1	0	0	5	22
-1	1	3	7	-7	-1	2	-3	-7	-1	3
-1	1	7	-5	-3	-5	-6	5	1	-1	3
-1	3	-7	-7	1	-3	6	-9	7	1	9
-1	3	-3	1	9	-3	2	-5	7	-11	-15
-1	3	-1	-5	-5	-9	6	3	-1	-5	15
-1	3	1	5	5	9	-6	3	-1	5	-15
-1	3	3	-1	-9	3	-2	-5	7	11	15
-1	3	7	7	-1	3	-6	-9	7	-1	-9
0	0	-6	-6	-8	8	-6	-2	-2	10	18
0	0	-6	-6	8	-8	-6	-2	-2	10	18
0	0	-6	-6	8	8	6	2	2	10	18
0	0	-3	-3	-4	-4	9	-1	-1	-2	6

Table 4: a_L values for newforms of level $\mathfrak{P}_2^4\mathfrak{P}_{13}^2$ (cont.)

$a_{L_3^0}$	$a_{L_3^1}$	$a_{L_{17}^0}$	$a_{L_{17}^1}$	$a_{L_{23}^0}$	$a_{L_{23}^1}$	a_{L_5}	$a_{L_{29}^0}$	$a_{L_{29}^1}$	a_{L_7}	$a_{L_{11}}$
0	0	-3	-3	-4	4	-9	1	1	-2	6
0	0	-3	-3	4	-4	-9	1	1	-2	6
0	0	3	3	-4	4	9	1	1	2	-6
0	0	3	3	4	-4	9	1	1	2	-6
0	0	3	3	4	4	-9	-1	-1	2	-6
0	0	6	6	-8	-8	-6	2	2	-10	-18
0	0	6	6	-8	8	6	-2	-2	-10	-18
0	0	6	6	8	-8	6	-2	-2	-10	-18
1	-3	-3	1	-9	3	2	-5	7	-11	-15
1	-3	-1	-5	5	9	6	3	-1	-5	15
1	-3	-1	7	-2	-2	-7	-8	0	1	6
1	-3	1	-7	2	2	7	-8	0	-1	-6
1	-3	1	5	-5	-9	-6	3	-1	5	-15
1	-3	3	-1	9	-3	-2	-5	7	11	15
1	-1	-7	-3	1	7	-2	-7	-3	1	-3
1	-1	-5	7	-5	-3	-6	1	5	-1	3
1	-1	-3	-3	-4	4	9	6	6	11	-18
1	-1	-3	-3	-3	3	10	9	9	-11	5
1	-1	-3	-3	0	0	1	-6	-6	-13	14
1	-1	-3	-3	1	-1	-6	-3	-3	13	21
1	-1	-3	-3	6	-6	1	0	0	-5	-22
1	-1	3	3	-6	6	-1	0	0	5	22
1	-1	3	3	-1	1	6	-3	-3	-13	-21
1	-1	3	3	0	0	-1	-6	-6	13	-14
1	-1	3	3	3	-3	-10	9	9	11	-5
1	-1	3	3	4	-4	-9	6	6	-11	18
1	-1	5	-7	5	3	6	1	5	1	-3
1	-1	7	3	-1	-7	2	-7	-3	-1	3
1	1	-7	-3	-1	7	2	7	3	1	-3
1	1	-7	5	3	-5	-6	-5	-1	1	-3
1	1	-5	7	5	-3	6	-1	-5	-1	3
1	1	-3	-7	7	-1	2	3	7	1	-3

Table 5: a_L values for newforms of level $\mathfrak{P}_2^4\mathfrak{P}_{13}^2$ (cont.)

$a_{L_3^0}$	$a_{L_3^1}$	$a_{L_{17}^0}$	$a_{L_{17}^1}$	$a_{L_{23}^0}$	$a_{L_{23}^1}$	a_{L_5}	$a_{L_{29}^0}$	$a_{L_{29}^1}$	a_{L_7}	$a_{L_{11}}$
1	1	-3	-3	-6	-6	-1	0	0	-5	-22
1	1	-3	-3	3	3	-10	-9	-9	-11	5
1	1	3	3	-3	-3	10	-9	-9	11	-5
1	1	3	3	6	6	1	0	0	5	22
1	1	3	7	-7	1	-2	3	7	-1	3
1	1	5	-7	-5	3	-6	-1	-5	1	-3
1	1	7	-5	-3	5	6	-5	-1	-1	3
1	1	7	3	1	-7	-2	7	3	-1	3
1	3	-7	-7	1	3	-6	9	-7	1	9
1	3	-3	1	9	3	-2	5	-7	-11	-15
1	3	-1	-5	-5	9	-6	-3	1	-5	15
1	3	-1	7	2	-2	7	8	0	1	6
1	3	1	-7	-2	2	-7	8	0	-1	-6
1	3	1	5	5	-9	6	-3	1	5	-15
1	3	3	-1	-9	-3	2	5	-7	11	15
1	3	7	7	-1	-3	6	9	-7	-1	-9
2	-2	-3	-3	6	-6	-7	-3	-3	14	22
2	-2	-3	-3	6	-6	-1	-9	-9	-2	22
2	-2	3	3	-6	6	1	-9	-9	2	-22
2	-2	3	3	-6	6	7	-3	-3	-14	-22
2	2	-3	-3	-6	-6	1	9	9	-2	22
2	2	3	3	6	6	-1	9	9	2	-22
3	-3	-3	-3	-4	4	9	-2	-2	-13	-18
3	-3	-3	-3	-1	1	-6	1	1	13	-3
3	-3	3	3	1	-1	6	1	1	-13	3
3	-3	3	3	4	-4	-9	-2	-2	13	18
3	-1	-7	-7	-3	1	6	7	-9	1	9
3	-1	-5	-1	-9	-5	6	-1	3	-5	15
3	-1	-1	3	3	-9	-2	7	-5	11	15
3	-1	1	-3	-3	9	2	7	-5	-11	-15
3	-1	5	1	9	5	-6	-1	3	5	-15

Table 6: a_L values for newforms of level $\mathfrak{P}_2^4\mathfrak{P}_{13}^2$ (cont.)

$a_{L_3^0}$	$a_{L_3^1}$	$a_{L_{17}^0}$	$a_{L_{17}^1}$	$a_{L_{23}^0}$	$a_{L_{23}^1}$	a_{L_5}	$a_{L_{29}^0}$	$a_{L_{29}^1}$	a_{L_7}	$a_{L_{11}}$
3	-1	7	7	3	-1	-6	7	-9	-1	-9
3	1	-7	-7	3	1	-6	-7	9	1	9
3	1	-7	1	2	-2	-7	0	8	-1	-6
3	1	-5	-1	9	-5	-6	1	-3	-5	15
3	1	-1	3	-3	-9	2	-7	5	11	15
3	1	1	-3	3	9	-2	-7	5	-11	-15
3	1	5	1	-9	5	6	1	-3	5	-15
3	1	7	-1	-2	2	7	0	8	1	6
3	1	7	7	-3	-1	6	-7	9	-1	-9
3	3	-3	-3	4	4	-9	2	2	-13	-18
3	3	3	3	-4	-4	9	2	2	13	18
-1	-1	-3	-3	-4	-4	-9	-6	-6	11	-18
-1	-1	-3	-3	0	0	-1	6	6	-13	14

Table 7: a_L values for newforms of level $\mathfrak{P}_2^4\mathfrak{P}_{13}^2$ (cont.)

6.2 B: Factorization of p_3

The polynomial p_3 on $S_2(\mathfrak{P}_2^4\mathfrak{P}_{13}^2)$ has the following factors:

$[x-3,38], [x-2,16], [x-1,84], [x,32], [x+1,92], [x+2,12], [x+3,45][x^2-4x+2,6], [x^2-3x-1,17], [x^2-3x+1,39], [x^2-2x-2,4], [x^2-2x-1,10], [x^2-x-5,18], [x^2-x-4,23], [x^2-x-3,33], [x^2-x-1,25], [x^2-8,10], [x^2-5,12], [x^2-3,20], [x^2-2,10], [x^2+x-5,6], [x^2+x-4,22], [x^2+x-3,42], [x^2+x-1,30], [x^2+2x-2,10], [x^2+2x-1,4], [x^2+3x-1,21], [x^2+3x+1,37], [x^2+4x+2,4], [x^3-4x^2+3x+1,4], [x^3-3x^2-4x+13,10], [x^3-2x^2-4x+7,6], [x^3-2x^2-x+1,4], [x^3-7x-7,4], [x^3-7x+7,8], [x^3-4x-1,9], [x^3-4x+1,6], [x^3+2x^2-4x-7,9], [x^3+2x^2-x-1,12], [x^3+3x^2-4x-13,4], [x^3+4x^2+3x-1,6], [x^4-4x^3-x^2+10x+2,6], [x^4-4x^3+8x-1,4], [x^4-3x^3-6x^2+23x-13,4], [x^4-2x^3-9x^2+22x-11,12], [x^4-2x^3-7x^2+8x-1,6], [x^4-2x^3-7x^2+8x+4,4], [x^4-2x^3-6x^2+10x+1,14], [x^4-2x^3-4x^2+8x-2,4], [x^4+2x^3-9x^2-22x-11,6], [x^4+2x^3-7x^2-8x-1,9], [x^4+2x^3-7x^2-8x+4,6], [x^4+2x^3-6x^2-10x+1,8], [x^4+2x^3-4x^2-8x-2,10], [x^4+3x^3-6x^2-23x-13,6], [x^4+4x^3-x^2-10x+2,4], [x^4+4x^3-8x-1,8], [x^6-3x^5-12x^4+36x^3+18x^2-68x+29,10], [x^6-3x^5-11x^4+31x^3+15x^2-25x-4,9], [x^6-3x^5-9x^4+23x^3+15x^2-13x-1,6], [x^6-2x^5-11x^4+16x^3+35x^2-26x-25,4], [x^6-x^5-13x^4+11x^3+49x^2-27x-52,9], [x^6+x^5-13x^4-11x^3+49x^2+27x-52,6], [x^6+2x^5-11x^4-16x^3+35x^2+26x-25,8], [x^6+3x^5-12x^4-36x^3+18x^2+68x+29,4], [x^6+3x^5-11x^4-31x^3+15x^2+25x-4,6], [x^6+3x^5-9x^4-23x^3+15x^2+13x-1,4], [x^8-6x^7+x^6+48x^5-65x^4-54x^3+115x^2-48x+4,8], [x^8-3x^7-20x^6+57x^5+124x^4-327x^3-245x^2+588x+16,4], [x^8+3x^7-20x^6-57x^5+124x^4+327x^3-245x^2-588x+16,2], [x^8+6x^7+x^6-48x^5-65x^4+54x^3+115x^2+48x+4,4], [x^9-4x^8-9x^7+50x^6-5x^5-156x^4+125x^3+50x^2-40x+4,6], [x^9-3x^8-11x^7+32x^6+38x^5-100x^4-47x^3+75x^2+37x+4,4], [x^9-2x^8-17x^7+34x^6+75x^5-158x^4-31x^3+106x^2-20x-4,4], [x^9-x^8-19x^7+16x^6+114x^5-76x^4-251x^3+165x^2+181x-128,6], [x^9+x^8-19x^7-16x^6+114x^5+76x^4-251x^3-165x^2+181x+128,4], [x^9+2x^8-17x^7-34x^6+75x^5+158x^4-31x^3-106x^2-20x+4,6], [x^9+3x^8-11x^7-32x^6+38x^5+100x^4-47x^3-75x^2+$

$37x-4,6],[x^9+4x^8-9x^7-50x^6-5x^5+156x^4+125x^3-50x^2-40x-4,4],[x^{12}-$
$x^{11}-22x^{10}+30x^9+153x^8-276x^7-317x^6+863x^5-182x^4-513x^3+242x^2+22x-$
$1,4],[x^{12}+x^{11}-22x^{10}-30x^9+153x^8+276x^7-317x^6-863x^5-182x^4+513x^3+$
$242x^2-22x-1,8],[x^{16}-2x^{15}-26x^{14}+48x^{13}+261x^{12}-442x^{11}-1300x^{10}+$
$2024x^9+3449x^8-4958x^7-4874x^6+6520x^5+3355x^4-4294x^3-776x^2+1104x-$
$74,6],[x^{16}+2x^{15}-26x^{14}-48x^{13}+261x^{12}+442x^{11}-1300x^{10}-2024x^9+3449x^8+$
$4958x^7-4874x^6-6520x^5+3355x^4+4294x^3-776x^2-1104x-74,4],[x^{18}-6x^{17}-$
$22x^{16}+184x^{15}+101x^{14}-2206x^{13}+1048x^{12}+13080x^{11}-12983x^{10}-39490x^9+$
$52906x^8+54352x^7-91701x^6-23122x^5+56412x^4+5600x^3-13034x^2-1480x+$
$568,4],[x^{18}-4x^{17}-33x^{16}+150x^{15}+387x^{14}-2233x^{13}-1578x^{12}+16799x^{11}-$
$4197x^{10}-65971x^9+59322x^8+117310x^7-188308x^6-21264x^5+190984x^4-$
$132221x^3+32604x^2-1200x-379,6],[x^{18}-3x^{17}-28x^{16}+82x^{15}+318x^{14}-$
$915x^{13}-1870x^{12}+5426x^{11}+5939x^{10}-18603x^9-9152x^8+37297x^7+2528x^6-$
$41228x^5+10028x^4+20669x^3-9779x^2-1970x+1259,4],[x^{18}-3x^{17}-28x^{16}+$
$84x^{15}+308x^{14}-921x^{13}-1692x^{12}+4994x^{11}+4927x^{10}-13943x^9-7648x^8+$
$19011x^7+6382x^6-10700x^5-3212x^4+2003x^3+627x^2-20x-1,4],[x^{18}-3x^{17}-$
$26x^{16}+78x^{15}+270x^{14}-815x^{13}-1436x^{12}+4416x^{11}+4153x^{10}-13377x^9-6258x^8+$
$22693x^7+3772x^6-20184x^5+782x^4+7699x^3-1277x^2-514x+13,4],[x^{18}-3x^{17}-$
$26x^{16}+80x^{15}+264x^{14}-837x^{13}-1362x^{12}+4500x^{11}+3737x^{10}-13433x^9-4682x^8+$
$22123x^7-202x^6-18304x^5+5878x^4+5249x^3-3531x^2+688x-43,4],[x^{18}-$
$2x^{17}-34x^{16}+64x^{15}+461x^{14}-802x^{13}-3208x^{12}+4968x^{11}+12385x^{10}-$
$15862x^9-26994x^8+24760x^7+31587x^6-14558x^5-16140x^4-640x^3+910x^2+$
$56x-8,6],[x^{18}+2x^{17}-34x^{16}-64x^{15}+461x^{14}+802x^{13}-3208x^{12}-4968x^{11}+$
$12385x^{10}+15862x^9-26994x^8-24760x^7+31587x^6+14558x^5-16140x^4+640x^3+$
$910x^2-56x-8,4],[x^{18}+3x^{17}-28x^{16}-84x^{15}+308x^{14}+921x^{13}-1692x^{12}-$
$4994x^{11}+4927x^{10}+13943x^9-7648x^8-19011x^7+6382x^6+10700x^5-3212x^4-$
$2003x^3+627x^2+20x-1,6],[x^{18}+3x^{17}-28x^{16}-82x^{15}+318x^{14}+915x^{13}-$
$1870x^{12}-5426x^{11}+5939x^{10}+18603x^9-9152x^8-37297x^7+2528x^6+41228x^5+$
$10028x^4-20669x^3-9779x^2+1970x+1259,8],[x^{18}+3x^{17}-26x^{16}-80x^{15}+$
$264x^{14}+837x^{13}-1362x^{12}-4500x^{11}+3737x^{10}+13433x^9-4682x^8-22123x^7-$
$202x^6+18304x^5+5878x^4-5249x^3-3531x^2-688x-43,6],[x^{18}+3x^{17}-26x^{16}-$
$78x^{15}+270x^{14}+815x^{13}-1436x^{12}-4416x^{11}+4153x^{10}+13377x^9-6258x^8-$
$22693x^7+3772x^6+20184x^5+782x^4-7699x^3-1277x^2+514x+13,6],[x^{18}+$
$4x^{17}-33x^{16}-150x^{15}+387x^{14}+2233x^{13}-1578x^{12}-16799x^{11}-4197x^{10}+$
$65971x^9+59322x^8-117310x^7-188308x^6+21264x^5+190984x^4+132221x^3+$
$32604x^2+1200x-379,4],[x^{18}+6x^{17}-22x^{16}-184x^{15}+101x^{14}+2206x^{13}+$
$1048x^{12}-13080x^{11}-12983x^{10}+39490x^9+52906x^8-54352x^7-91701x^6+$
$23122x^5+56412x^4-5600x^3-13034x^2+1480x+568,6],[x^{27}-3x^{26}-55x^{25}+$
$164x^{24}+1304x^{23}-3858x^{22}-17519x^{21}+51323x^{20}+147499x^{19}-427049x^{18}-$
$812192x^{17}+2322310x^{16}+2957083x^{15}-8374150x^{14}-7010721x^{13}+19892990x^{12}+$
$10323950x^{11}-30317349x^{10}-8509315x^9+28202264x^8+2963409x^7-14792310x^6+$
$220229x^5+3889959x^4-397376x^3-380960x^2+72460x-2647,6],[x^{27}-3x^{26}-$
$53x^{25}+160x^{24}+1210x^{23}-3680x^{22}-15631x^{21}+47937x^{20}+126405x^{19}-$
$390929x^{18}-670268x^{17}+2085994x^{16}+2381831x^{15}-7410738x^{14}-5717857x^{13}+$
$17541450x^{12}+9224648x^{11}-27292133x^{10}-9719725x^9+27067090x^8+6187223x^7-$
$16180684x^6-1937401x^5+5264635x^4+99800x^3-738836x^2+47264x+14987,6],[x^{27}+$
$3x^{26}-55x^{25}-164x^{24}+1304x^{23}+3858x^{22}-17519x^{21}-51323x^{20}+147499x^{19}+$
$427049x^{18}-812192x^{17}-2322310x^{16}+2957083x^{15}+8374150x^{14}-7010721x^{13}-$
$19892990x^{12}+10323950x^{11}+30317349x^{10}-8509315x^9-28202264x^8+2963409x^7+$
$14792310x^6+220229x^5-3889959x^4-397376x^3+380960x^2+72460x+2647,4],[x^{27}+$

$3x^{26}-53x^{25}-160x^{24}+1210x^{23}+3680x^{22}-15631x^{21}-47937x^{20}+126405x^{19}+390929x^{18}-670268x^{17}-2085994x^{16}+2381831x^{15}+7410738x^{14}-5717857x^{13}-17541450x^{12}+9224648x^{11}+27292133x^{10}-9719725x^{9}-27067090x^{8}+6187223x^{7}+16180684x^{6}-1937401x^{5}-5264635x^{4}+99800x^{3}+738836x^{2}+47264x-14987, 4]$

The polynomial p_3 on $S_2(\mathfrak{P}_2^3\mathfrak{P}_{13}^2)$ has the following factors:

$[x-3, 8], [x-2, 8], [x-1, 20], [x, 12], [x+1, 28], [x+2, 4], [x+3, 15], [x^2-4x+2, 2], [x^2-3x-1, 3], [x^2-3x+1, 11], [x^2-2x-1, 6], [x^2-x-5, 12], [x^2-x-4, 5], [x^2-x-3, 3], [x^2-x-1, 5], [x^2-8, 2], [x^2-5, 4], [x^2-3, 4], [x^2-2, 2], [x^2+x-4, 4], [x^2+x-3, 12], [x^2+x-1, 10], [x^2+2x-2, 6], [x^2+3x-1, 7], [x^2+3x+1, 9], [x^3-3x^2-4x+13, 6], [x^3-7x+7, 4], [x^3-4x-1, 3], [x^3+2x^2-4x-7, 3], [x^3+2x^2-x-1, 8], [x^3+4x^2+3x-1, 2], [x^4-4x^3-x^2+10x+2, 2], [x^4-2x^3-9x^2+22x-11, 6], [x^4-2x^3-6x^2+10x+1, 6], [x^4+2x^3-7x^2-8x-1, 3], [x^4+2x^3-7x^2-8x+4, 2], [x^4+2x^3-4x^2-8x-2, 6], [x^4+3x^3-6x^2-23x-13, 2], [x^4+4x^3-8x-1, 4], [x^6-3x^5-12x^4+36x^3+18x^2-68x+29, 6], [x^6-3x^5-11x^4+31x^3+15x^2-25x-4, 3], [x^6-3x^5-9x^4+23x^3+15x^2-13x-1, 2], [x^6-x^5-13x^4+11x^3+49x^2-27x-52, 3], [x^6+2x^5-11x^4-16x^3+35x^2+26x-25, 4], [x^8-6x^7+x^6+48x^5-65x^4-54x^3+115x^2-48x+4, 4], [x^8-3x^7-20x^6+57x^5+124x^4-327x^3-245x^2+588x+16, 2], [x^9-4x^8-9x^7+50x^6-5x^5-156x^4+125x^3+50x^2-40x+4, 2], [x^9-x^8-19x^7+16x^6+114x^5-76x^4-251x^3+165x^2+181x-128, 2], [x^9+2x^8-17x^7-34x^6+75x^5+158x^4-31x^3-106x^2-20x+4, 2], [x^9+3x^8-11x^7-32x^6+38x^5+100x^4-47x^3-75x^2+37x-4, 2], [x^{12}+x^{11}-22x^{10}-30x^9+153x^8+276x^7-317x^6-863x^5-182x^4+513x^3+242x^2-22x-1, 4], [x^{16}-2x^{15}-26x^{14}+48x^{13}+261x^{12}-442x^{11}-1300x^{10}+2024x^9+3449x^8-4958x^7-4874x^6+6520x^5+3355x^4-4294x^3-776x^2+1104x-74, 2], [x^{18}-4x^{17}-33x^{16}+150x^{15}+387x^{14}-2233x^{13}-1578x^{12}+16799x^{11}-4197x^{10}-65971x^9+59322x^8+117310x^7-188308x^6-21264x^5+190984x^4-132221x^3+32604x^2-1200x-379, 2], [x^{18}-2x^{17}-34x^{16}+64x^{15}+461x^{14}-802x^{13}-3208x^{12}+4968x^{11}+12385x^{10}-15862x^9-26994x^8+24760x^7+31587x^6-14558x^5-16140x^4-640x^3+910x^2+56x-8, 2], [x^{18}+3x^{17}-28x^{16}-84x^{15}+308x^{14}+921x^{13}-1692x^{12}-4994x^{11}+4927x^{10}+13943x^9-7648x^8-19011x^7+6382x^6+10700x^5-3212x^4-2003x^3+627x^2+20x-1, 2], [x^{18}+3x^{17}-28x^{16}-82x^{15}+318x^{14}+915x^{13}-1870x^{12}-5426x^{11}+5939x^{10}+18603x^9-9152x^8-37297x^7+2528x^6+41228x^5+10028x^4-20669x^3-9779x^2+1970x+1259, 4], [x^{18}+3x^{17}-26x^{16}-80x^{15}+264x^{14}+837x^{13}-1362x^{12}-4500x^{11}+3737x^{10}+13433x^9-4682x^8-22123x^7-202x^6+18304x^5+5878x^4-5249x^3-3531x^2-688x-43, 2], [x^{18}+3x^{17}-26x^{16}-78x^{15}+270x^{14}+815x^{13}-1436x^{12}-4416x^{11}+4153x^{10}+13377x^9-6258x^8-22693x^7+3772x^6+20184x^5+782x^4-7699x^3-1277x^2+514x+13, 2], [x^{18}+6x^{17}-22x^{16}-184x^{15}+101x^{14}+2206x^{13}+1048x^{12}-13080x^{11}-12983x^{10}+39490x^9+52906x^8-54352x^7-91701x^6+23122x^5+56412x^4-5600x^3-13034x^2+1480x+568, 2], [x^{27}-3x^{26}-55x^{25}+164x^{24}+1304x^{23}-3858x^{22}-17519x^{21}+51323x^{20}+147499x^{19}-427049x^{18}-812192x^{17}+2322310x^{16}+2957083x^{15}-8374150x^{14}-7010721x^{13}+19892990x^{12}+10323950x^{11}-30317349x^{10}-8509315x^9+28202264x^8+2963409x^7-14792310x^6+220229x^5+3889959x^4-397376x^3-380960x^2+72460x-2647, 2], [x^{27}-3x^{26}-53x^{25}+160x^{24}+1210x^{23}-3680x^{22}-15631x^{21}+47937x^{20}+126405x^{19}-390929x^{18}-670268x^{17}+2085994x^{16}+2381831x^{15}-7410738x^{14}-5717857x^{13}+17541450x^{12}+9224648x^{11}-27292133x^{10}-9719725x^9+27067090x^8+6187223x^7-16180684x^6-1937401x^5+5264635x^4+99800x^3-738836x^2+47264x+14987, 2]$

References

[1] N. Billerey. Équations de Fermat de Type $(5, 5, p)$. *Bull. Austral. Math. Soc.*, 76(2):161–194, 2007.

[2] N. Billerey and L. Dieulefait. Solving Fermat-type equations $x^5 + y^5 = dz^p$. *Math. Comp.*, 79:535–544, 2010.

[3] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24:235–265, 1997.

[4] C. Breuil. p-adic hodge theory, deformations and local langlands. *notes of a course at CRM, Bellaterra, Spain, July 2001.*

[5] N. Bruin. On powers as sums of two cubes. *Algorithmic number theory (edited by W. Bosma), Lecture Notes in Comput. Sci. 1838, Springer, Berlin.*

[6] I. Chen and M. Bennett. Multi-frey $\mathbb{Q}$-curves and the diophantine equation $a^2 + b^6 = c^p$. *http://people.math.sfu.ca/ ichen/pub/BeCh2.pdf.*

[7] I. Chen and S. Siksek. Perfect powers expressible as sums of two cubes. *J. Algebra*, 322:638–656, 2009.

[8] S. Dahmen. Classical and modular methods applied to diophantine equations. *PhD thesis, University of Utrecht, avaliable at igitur-archive.library.uu.nl/dissertations/2008-0820-200949/UUindex.html*, 2008.

[9] H. Darmon and A. Granville. On the equations $z^m = f(x, y)$ and $ax^p + by^q = cz^r$. *Bull. of London Math. Soc.*, 27:513–543, 1995.

[10] L. Dembélé and J. Voight. Explicit methods for hilbert modular forms. *Elliptic curves, Hilbert modular forms and Galois deformations. Diamond et al. (eds.). Birkhauser, Progress in Mathematics, to appear.*

[11] L. Dieulefait. Modularity of abelian surfaces with quaternionic multiplication. *Math. Research. Letters*, 10:145–150, 2003.

[12] L. Dieulefait and N. Freitas. The Fermat-type equations $x^5 + y^5 = 2z^p$ or $3z^p$ solved through $\mathbb{Q}$-curves. *submited to publication.*

[13] J. Ellenberg. Galois representations attached to $\mathbb{Q}$-curves and the generalized Fermat equation $A^4 + B^2 = C^p$. *Amer. J. Math*, 126:763–787, 2004.

[14] J. Ellenberg. Serre conjecture over $\mathbb{F}_9$. *Ann. of Math*, 161:1111–1142, 2005.

[15] K. Fujiwara. Level optimisation in the totally real case, preprint. 2006.

[16] F. Jarvis. Correspondences on Shimura curves and Mazur's Principle at p. *Pacific J. Math.*, 213:267–280, 2004.

[17] F. Jarvis and J. Manoharmayum. On the modularity of supersingular elliptic curves over certain totally real number fields. *J. Number Theory*, 128 (3), 2008.

[18] F. Jarvis and P. Meekin. The Fermat equation over $\mathbb{Q}(\sqrt{2})$. *Journal of Number Theory*, 537:182–196, 2004.

[19] M. Kisin. Modularity of 2-adic galois representations. *Current Developments in Mathematics*, pages 191–230, 2005.

[20] A. Kraus. Sur l'équation $a^3 + b^3 = c^p$. *Experiment. Math.*, 7:1–13, 1998.

[21] J. Manoharmayum. On the modularity of certain $\mathrm{GL}(2, \mathbb{F}_7)$ Galois representations. *Math. Res Letters*, 8:703–712, 2001.

[22] I. Papadopoulos. Sur la classification de néron des courbes elliptiques en caractéristique résiduelle 2 et 3. *Journal of Number Theory*, 44:119–152, 1993.

[23] A. Rajaei. On the levels of mod Hilbert modular forms. *J. reine angew.*, 537:33–65, 2001.

[24] C. Skinner and A. Wiles. Residually reducible representations and modular forms. *Publ. Math. IHES*, 89:5–126, 2000.

[25] C. Skinner and A. Wiles. Nearly ordinary deformations of irreducible residual representations. *Annales de la faculté des sciences de Toulouse*, (6)10:185–215, 2001.

[26] R. Taylor. On icosaedral Artin representations ii. *American J. Math.*, 125:549–566, 2003.

[27] A. Wiles. Modular elliptic curves and fermat's last theorem. *Ann. of Math*, 141, 1995.